9.3 The Dot Product Notes

Definition and Algebraic Properties of the Dot Product

We begin with the following definition.

Definition. Suppose $\mathbf{v}$ and $\mathbf{w}$ are vectors whose component forms are $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$. The **dot product** of $\mathbf{v}$ and $\mathbf{w}$ is given by

$$\mathbf{v} \cdot \mathbf{w} = \langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle = v_1 w_1 + v_2 w_2$$

Example 9.3.1. Find the dot product of $\mathbf{v} = \langle 3, 4 \rangle$ and $\mathbf{w} = \langle 1, -2 \rangle$.

Solution. We have

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= \langle 3, 4 \rangle \cdot \langle 1, -2 \rangle \\ &= (3)(1) + (4)(-2) \\ &= -5 \end{aligned}$$

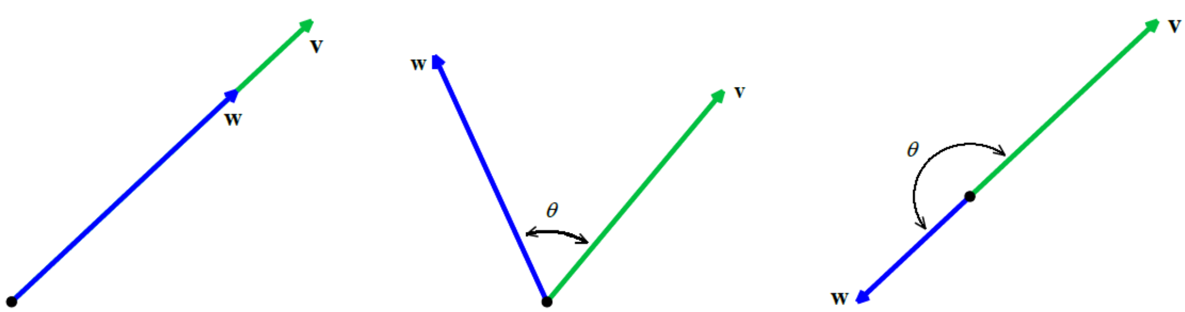
□

Theorem 9.5. Properties of the Dot Product:

- **Commutative Property:** For all vectors $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.
- **Distributive Property:** For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
- **Scalar Property:** For all vectors $\mathbf{v}$ and $\mathbf{w}$, and scalars k, $(k\mathbf{v}) \cdot \mathbf{w} = k(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (k\mathbf{w})$.
- **Relation to Magnitude:** For all vectors $\mathbf{v}$, $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.

Geometric Properties of the Dot Product

Suppose $\mathbf{v}$ and $\mathbf{w}$ are two nonzero vectors. If we draw $\mathbf{v}$ and $\mathbf{w}$ with the same initial point, we define the **angle between** $\mathbf{v}$ and $\mathbf{w}$ to be the angle θ determined by the rays containing the vectors $\mathbf{v}$ and $\mathbf{w}$, as illustrated below. We choose to define $0 \le \theta \le \pi$.



$\theta = 0$ $\qquad$ $0 < \theta < \pi$ $\qquad$ $\theta = \pi$

The following theorem gives us some insight into the geometric role the dot product plays.

Theorem 9.6. Geometric Interpretation of the Dot Product: If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors then $\mathbf{v}\cdot\mathbf{w}=\|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

Determining the Angle Between Two Vectors

An immediate consequence of **Theorem 9.6** is the following.

Theorem 9.7. Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors and let θ be the angle between $\mathbf{v}$ and $\mathbf{w}$. Then

$$\theta=\arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)$$

Example 9.3.3. Find the angle between the following pairs of vectors.

1. $\mathbf{v}=\left\langle 3,-3\sqrt{3}\right\rangle$ and $\mathbf{w}=\left\langle -\sqrt{3},1\right\rangle$

Solution. We use the formula $\theta=\arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)$ from **Theorem 9.7** in each case below.

1. For $\mathbf{v}=\left\langle 3,-3\sqrt{3}\right\rangle$ and $\mathbf{w}=\left\langle -\sqrt{3},1\right\rangle$, we have

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w}&=\left\langle 3,-3\sqrt{3}\right\rangle\cdot\left\langle -\sqrt{3},1\right\rangle\\&=-3\sqrt{3}-3\sqrt{3}\\&=-6\sqrt{3}\end{aligned}$$

$$\begin{aligned}\|\mathbf{v}\|&=\sqrt{3^2+\left(-3\sqrt{3}\right)^2}\\&=\sqrt{36}\\&=6\end{aligned}$$

$$\begin{aligned}\|\mathbf{w}\|&=\sqrt{\left(-\sqrt{3}\right)^2+1^2}\\&=\sqrt{4}\\&=2\end{aligned}$$

Then

$$\begin{aligned}\theta&=\arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)\\&=\arccos\left(\frac{-6\sqrt{3}}{12}\right)\\&=\arccos\left(-\frac{\sqrt{3}}{2}\right)\\&=\frac{5\pi}{6}\end{aligned}$$

Now Try:

Examples: Find the smallest positive angle between the following vectors:

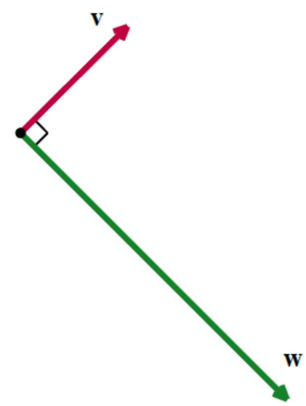
a) $\langle 1,3\rangle$ and $\langle 5,2\rangle$

b) $\langle -5,1\rangle$ and $\langle 7,3\rangle$

Orthogonal Vectors

The vectors $\mathbf{v}=\langle 2,2\rangle$ and $\mathbf{w}=\langle 5,-5\rangle$ are called **orthogonal**, and we write $\mathbf{v}\perp\mathbf{w}$, because the angle between them is $\frac{\pi}{2}$ radians, or 90°. Geometrically, when orthogonal vectors are sketched with the same initial point, the lines containing the vectors are perpendicular.

In the illustration to the right, $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, and we write $\mathbf{v}\perp\mathbf{w}$.

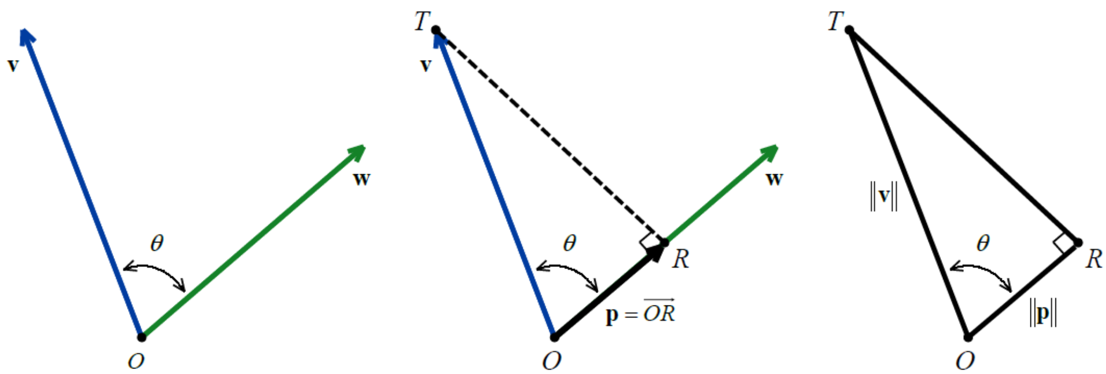


We state the relationship between orthogonal vectors and their dot product in the following theorem.

Theorem 9.8. The Dot Product Detects Orthogonality: Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors. Then $\mathbf{v}\perp\mathbf{w}$ if and only if $\mathbf{v}\cdot\mathbf{w}=0$.

Orthogonal Projection

Consider the two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ drawn with a common initial point O below. For the moment, assume that the angle between $\mathbf{v}$ and $\mathbf{w}$, which we'll denote θ, is acute. We wish to develop a formula for the vector $\mathbf{p}$, indicated below, which is called the **orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$**. The vector $\mathbf{p}$ is obtained geometrically as follows: drop a perpendicular from the terminal point T of $\mathbf{v}$ to the vector $\mathbf{w}$ and call the point of intersection R. The vector $\mathbf{p}$ is then defined as $\mathbf{p}=\overrightarrow{OR}$.



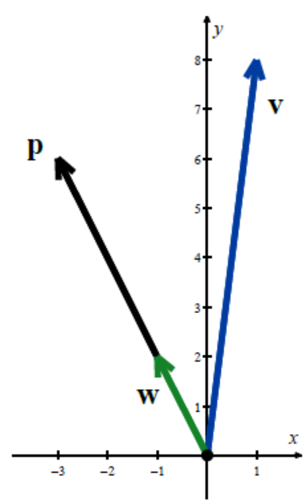
Theorem 9.9. If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors, then the **orthogonal projection of** $\mathbf{v}$ **onto** $\mathbf{w}$, denoted $\text{proj}_{\mathbf{w}}(\mathbf{v})$, is given by

$$\text{proj}_{\mathbf{w}}(\mathbf{v}) = \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w}$$

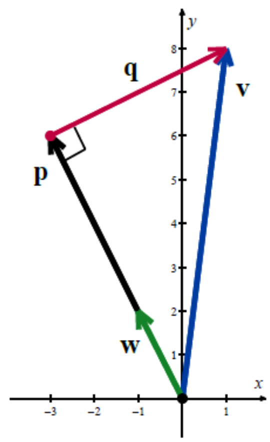
Example 9.3.4. Let $\mathbf{v} = \langle 1,8\rangle$ and $\mathbf{w} = \langle -1,2\rangle$. Find $\mathbf{p} = \text{proj}_{\mathbf{w}}(\mathbf{v})$, and plot $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{p}$ in standard position.

$$\begin{aligned}
\text{proj}_{\mathbf{w}}(\mathbf{v}) &= \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} \\
&= \left(\frac{\langle 1,8\rangle\cdot\langle -1,2\rangle}{\|\langle -1,2\rangle\|^2}\right)\langle -1,2\rangle \\
&= \left(\frac{(1)(-1)+(8)(2)}{\left(\sqrt{(-1)^2+(2)^2}\right)^2}\right)\langle -1,2\rangle \\
&= \left(\frac{-1+16}{\left(\sqrt{5}\right)^2}\right)\langle -1,2\rangle \\
&= 3\langle -1,2\rangle
\end{aligned}$$

Hence, $\mathbf{p} = \text{proj}_{\mathbf{w}}(\mathbf{v}) = \langle -3,6\rangle$. We plot $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{p}$ below.



Suppose we wanted to verify that our answer $\mathbf{p}$ in **Example 9.3.4** is indeed the orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$. We first note that, since $\mathbf{p}=3\mathbf{w}$, $\mathbf{p}$ is a scalar multiple of $\mathbf{w}$ and so it has the correct direction. It remains to check the orthogonality condition. Consider the vector $\mathbf{q}$ whose initial point is the terminal point of $\mathbf{p}$ and whose terminal point is the terminal point of $\mathbf{v}$.



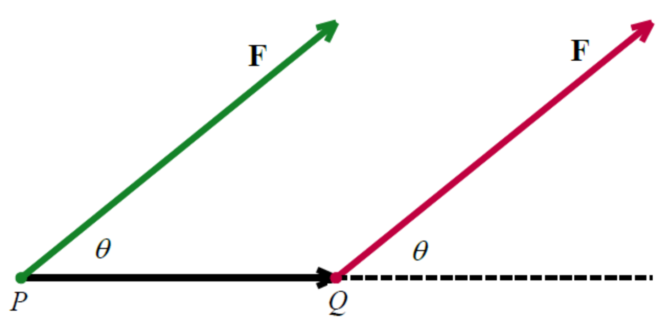
From the definition of vector arithmetic, $\mathbf{p}+\mathbf{q}=\mathbf{v}$, so that $\mathbf{q}=\mathbf{v}-\mathbf{p}$. In the case of **Example 9.3.4**, $\mathbf{v}=\langle 1,8\rangle$ and $\mathbf{p}=\langle -3,6\rangle$, so $\mathbf{q}=\langle 1,8\rangle-\langle -3,6\rangle=\langle 4,2\rangle$. Then

$$\begin{aligned}\mathbf{q}\cdot\mathbf{v}&=\langle 4,2\rangle\cdot\langle -1,2\rangle\\&=(-4)+4\\&=0\end{aligned}$$

This shows $\mathbf{q}\perp\mathbf{w}$ as required.

Work

We close this section with an application of the dot product. In Physics, if a constant force $\mathbf{F}$ moves an object a distance d, then the **work**, W, done by the force is given by the magnitude of the force times the amount of displacement, or $W=\|\mathbf{F}\|d$, where the force is being applied in the direction of the motion. If the force applied is not in the direction of the motion, we can use the dot product to find the work done. Consider the scenario below where the constant force $\mathbf{F}$ is applied to move an object from the point P to the point Q.



To determine the work W done in this scenario, we find that the magnitude of the force $\mathbf{F}$ in the direction of $\overrightarrow{PQ}$ is $\|\mathbf{F}\|\cos(\theta)$, where θ is the angle between $\mathbf{F}$ and $\overrightarrow{PQ}$. The distance the object travels is $\left\|\overrightarrow{PQ}\right\|$. Since work is the magnitude of the force $\mathbf{F}$ in the direction of $\overrightarrow{PQ}$ times the distance traveled from P to Q, we get

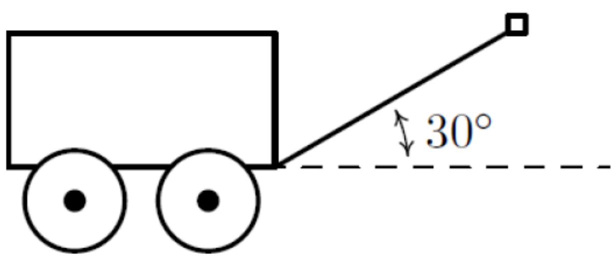
$$\begin{aligned} W &= \|\mathbf{F}\|\cos(\theta)\left\|\overrightarrow{PQ}\right\| \\ &= \|\mathbf{F}\|\left\|\overrightarrow{PQ}\right\|\cos(\theta) \\ &= \mathbf{F}\cdot\overrightarrow{PQ} \qquad \textbf{from Theorem 9.6} \end{aligned}$$

Theorem 9.10. Work as a Dot Product: Suppose a constant force $\mathbf{F}$ is applied to move an object along the vector $\overrightarrow{PQ}$, from P to Q. The work W done by $\mathbf{F}$ is given by

$$W = \mathbf{F}\cdot\overrightarrow{PQ} = \|\mathbf{F}\|\left\|\overrightarrow{PQ}\right\|\cos(\theta),$$

where θ is the angle between $\mathbf{F}$ and $\overrightarrow{PQ}$.

Example 9.3.5. Taylor exerts a force of 10 pounds to pull her wagon a distance of 50 feet over level ground. If the handle of the wagon makes a 30° angle with the horizontal, how much work did Taylor do pulling the wagon? Assume Taylor exerts the force of 10 pounds at a 30° angle for the duration of the 50 feet.



Solution. There are two ways to attack this problem.

- One way is to find the vectors $\mathbf{F}$ and $\overrightarrow{PQ}$ mentioned in **Theorem 9.10** and compute $W = \mathbf{F}\cdot\overrightarrow{PQ}$. To do this, we assume the origin is at the point where the handle of the wagon meets the wagon and the positive x-axis lies along the dashed line in the figure above. Since the force applied is a constant 10 pounds, we have $\|\mathbf{F}\| = 10$. Since it is being applied at a constant angle of $\theta = 30^\circ$ with respect to the positive x-axis, **Theorem 9.3** gives us

$$\begin{aligned} \mathbf{F} &= 10\left\langle \cos\left(30^\circ\right), \sin\left(30^\circ\right)\right\rangle \\ &= 10\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle \\ &= \left\langle 5\sqrt{3}, 5\right\rangle \end{aligned}$$

Since the wagon is being pulled along 50 feet in the positive direction, the displacement vector is

$$\begin{aligned}\overrightarrow{PQ} &= 50i \\ &= 50\langle 1,0\rangle \\ &= \langle 50,0\rangle\end{aligned}$$

$$\begin{aligned}W &= \mathbf{F}\cdot\overrightarrow{PQ} \\ &= \left\langle 5\sqrt{3},5\right\rangle\cdot\langle 50,0\rangle \\ &= 250\sqrt{3}\end{aligned}$$

Since force is measured in pounds and distance is measured in feet, we get $W = 250\sqrt{3}$ foot-pounds.

- Alternately, we can use the formulation $W = \|\mathbf{F}\|\left\|\overrightarrow{PQ}\right\|\cos(\theta)$ to get

$$\begin{aligned}W &= (10\text{ pounds})(50\text{ feet})\cos\left(30^\circ\right) \\ &= (500)\left(\frac{\sqrt{3}}{2}\right)\text{ foot-pounds} \\ &= 250\sqrt{3}\text{ foot-pounds of work}\end{aligned}$$